Last Time: Chromatic polynomial.

Recall: Chromatic polynomial of G is the function $\mathbb{N}_0 \to \mathbb{N}_0$.

$$P_G(k) := \#\{ c : V(G) \to [k] : c \text{ is a coloring of } G\}$$

Ex: $P_{K_n}(k) = k(k-1)\cdots(k-(n-1))$.

Prop: Given graph G and $e \in E(G)$ which is not a loop,

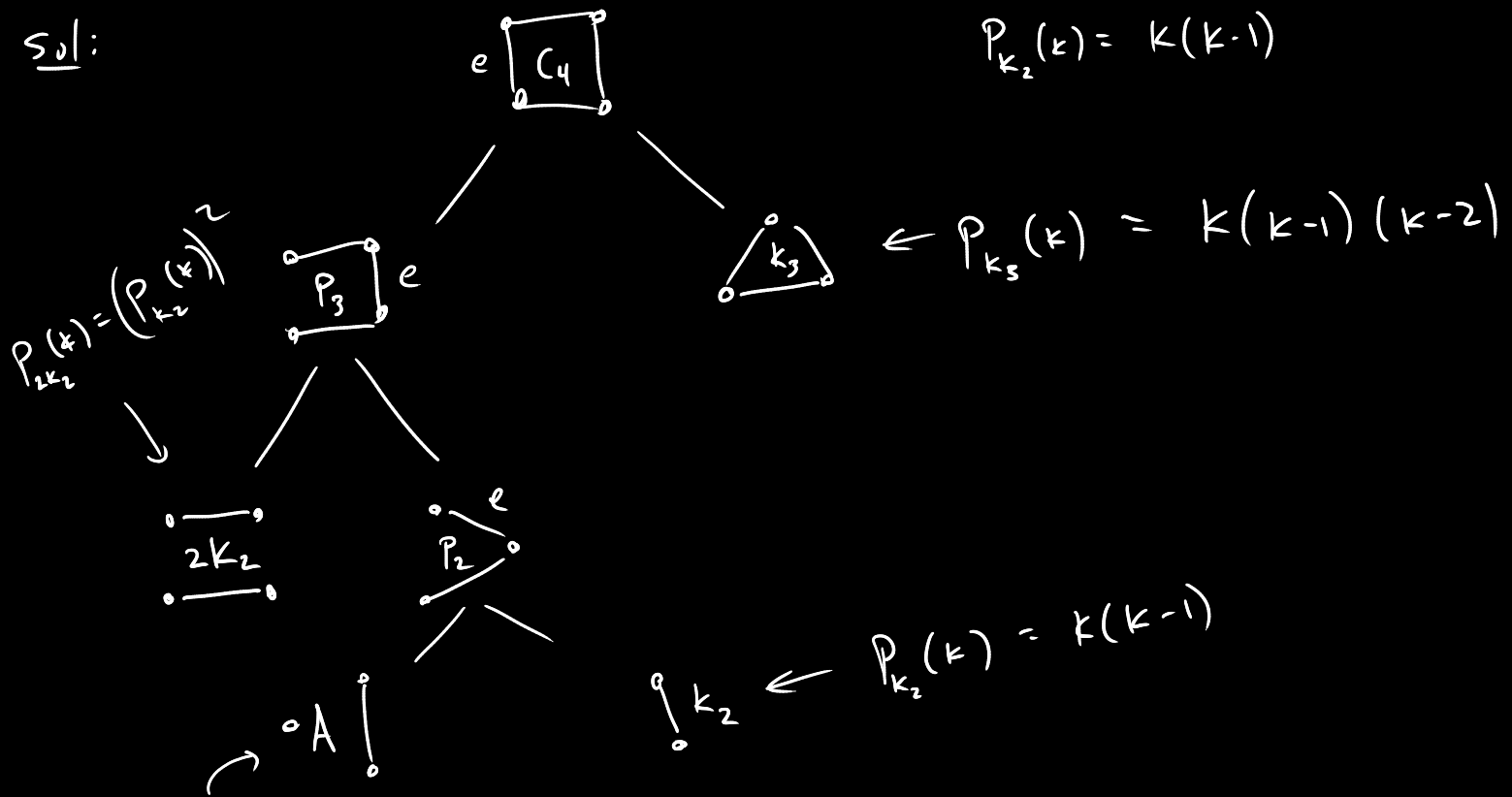
$$P_G(k) = P_{G\setminus e}(k) - P_{G/e}(k).$$

Prop: If G is a graph with components $G_1, G_2, \dots, G_m$, then

$$P_G(k) = \prod_{i=1}^{m} P_{G_i}(k).$$ pf: Exercise. ☒

Ex: Compute $P_{C_4}(k)$ using the previous two results.

Sol:



$$P_A(k) = P_{K_1}(k)\cdot P_{K_2}(k) = k\cdot k(k-1)$$

$$\therefore P_{C_4}(k) = P_{P_3}(k) - P_{K_3}(k) = \left(P_{2K_2}(k) - P_{P_2}(k)\right) - k(k-1)(k-2)$$

$$= \left(k(k-1)\right)^2 - \left(P_A(k) - P_{K_2}(k)\right) - k(k-1)(k-2)$$

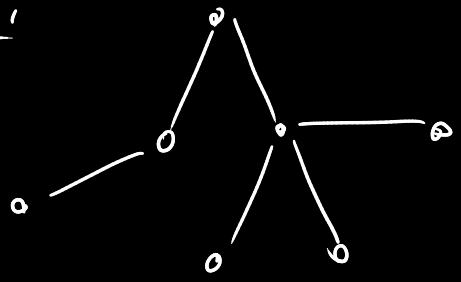
$$= k^2(k-1)^2 - k^2(k-1) + k(k-1) - k(k-1)(k-2)$$

$$= k(k-1)\left(k(k-1) - k + 1 - (k-2)\right) = k(k-1)(k^2-3k+3)$$ ★ ☒

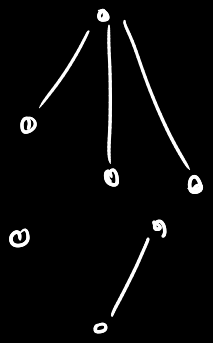
Trees and Forests

Defn: A graph G is *acyclic* when it does not have any cycles. An acyclic graph is called a *forest*. A *tree* is a connected forest (i.e. a tree is a connected acyclic graph).

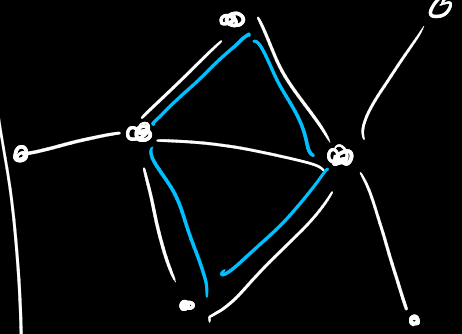
Ex:



T is a tree



F is a forest



G is *not* a forest

Observation: Every tree is a forest. (lonely forest...).

Q: What other conditions classify trees?

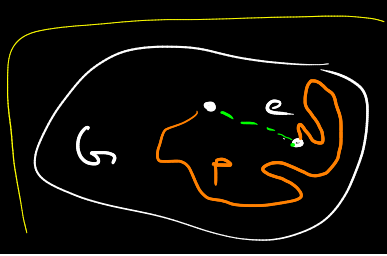
Prop: Let G be a graph. The following are equivalent.

1. G is a tree.
2. G is acyclic and connected.
3. G is minimally connected.
   (i.e. G is connected, but $G \setminus e$ is disconnected for all $e \in E(G)$)
4. G is maximally acyclic.
   (i.e. G is acyclic, but adding any new edge to G creates a cycle)

Pf: Let G be a graph.

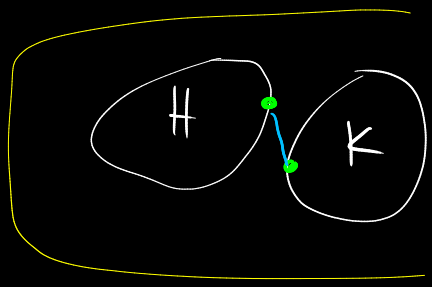
(1) $\Leftrightarrow$ (2): This is the definition of tree!

(2) $\Rightarrow$ (3): Suppose G is acyclic and connected. Assume to the contrary that $G \setminus e$ is connected for some $e \in E(G)$. Let P be any path in $G \setminus e$ connecting the ends of e. Now $P \cup e$ forms a cycle subgraph of G, contradicting the assumption that G is acyclic. Hence G is minimally connected.



③ ⇒ ④: Assume G is minimally connected. Assume to the contrary that we may add an edge e between vertices u, v so that $G \cup e$ is acyclic. Similar to before, there was a path P in G connecting u to v, so $G \cup e$ has subgraph $P \cup e$ which is necessarily a cycle in $G \cup e$, contradicting the assumption that $G \cup e$ was acyclic. Hence G is maximally acyclic.

④ ⇒ ②: Assume G is maximally acyclic. Assume to the contrary that G has at least two components, H and K. Choose any two vertices $u \in V(H)$ and $v \in V(K)$.



By assumption, no walk in G connects u to v (otherwise $H = K$ is still one component...).

So adding the edge e from u to v does not create a cycle in $G \cup e$; if it did, e must belong to the cycle. Walking around the cycle not including e would produce a walk from u to v in G contradicting our assumption these were in different components. Hence, under these assumptions we were able to add an edge e to G for which $G \cup e$ is acyclic; this contradicts our assumption that G is maximally acyclic!

Hence G is connected.

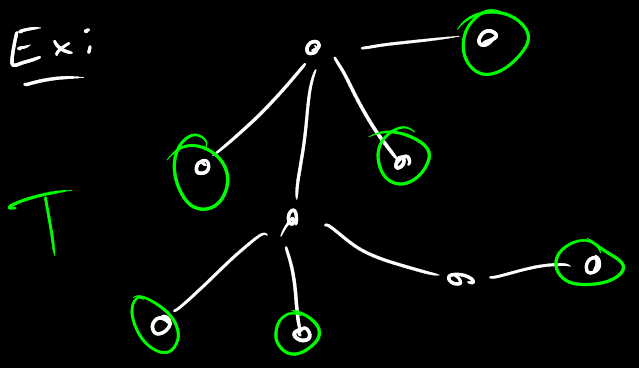
Hence we've proved ① ⇔ ② ⇒ ③ ⇒ ④ ⇒ ②, verifying their equivalence ☑

Prop: Graph G is a tree if and only if every pair of vertices in G is connected by a unique path. pf: Exercise ☑

Q: What kind of structural results can we prove about trees?

Defn: Let T be a tree. A leaf of T is a vertex of degree 1 in T.

Ex:



leaves are circled in

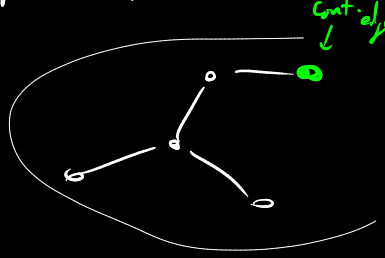
Prop: If T is a tree with at least two vertices, then T has at least two leaves.

pf: Let T be a tree with at least two vertices.

We proceed by induction on the number of edges of T.

Base case: If T has one edge, then T is the graph $u \overset{e}{\text{---}} v$. this graph T has two leaves (namely u and v).

Inductive Step: Suppose every tree with n edges has at least two leaves, and suppose T has $n+1$ edges. Choose any edge e of T. The contraction T/e is acyclic lest we could build a cycle in T from a cycle in T/e; T/e is also connected, lest T is not connected. Hence T/e is a tree with $(n+1)-1 = n$ edges. By the inductive hypothesis, T/e has at least two leaves. If both of the leaves obtained in this way are not ends of e in T, then they were leaves of T. Otherwise one of the leaves was the result of contraction; however, this implies one of the ends of e was a leaf of T. Indeed, if not, then both ends of e have degree at least 2 in T and hence have degree at least 2 in T/e. Hence T has at least two leaves in all cases.



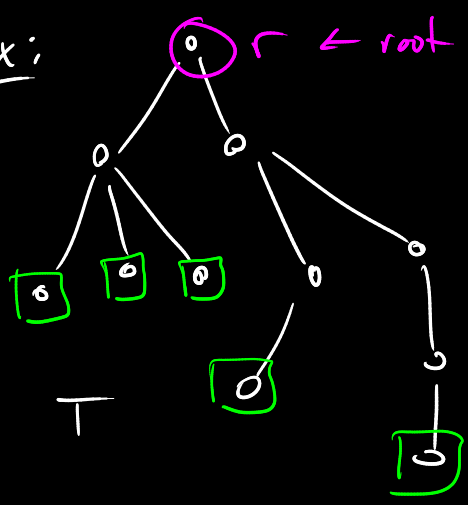
We conclude the original statement holds by weak math. induction. ☐

Remark: This proof can be adapted to show every tree with a vertex of degree d has at least d leaves...

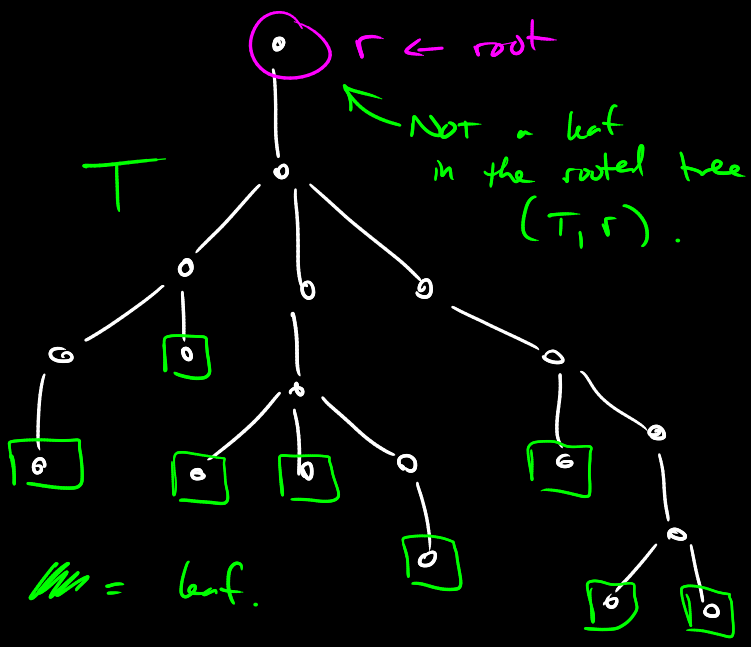
Defn: A root of tree T is a vertex which is "designated".

↳ **NB:** by convention, in a rooted tree (i.e. a pair (T, r) with T a tree and $r \in V(T)$) we don't call r a leaf.

Ex:

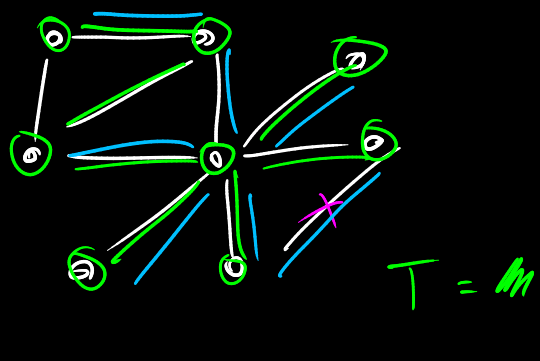


(T, r) is a rooted tree.

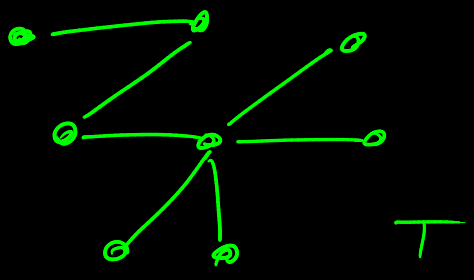


Defn: A spanning tree in connected graph G is a subgraph $T \leq G$ satisfying ① T is a tree and ② $V(T) = V(G)$.

Ex: A connected graph G often has MANY spanning trees.



G



Next time: Prim's algorithm for constructing "minimum weight" spanning trees in graph G.